

# ON CLASSIFYING HUREWICZ FIBRATIONS AND FIBRE BUNDLES OVER POLYHEDRON BASES

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## Abstract

Let  $f : E \longrightarrow O$  be a Hurewicz fibration with a fiber space  $F_{r_o}$  and a lifting function  $L_f$ . The  $Lf$ -function  $\Theta_{L_f}$  of  $f$  is defined by the restriction map of  $L_f$  on the space  $\Omega(O, r_o) \times F_{r_o} \times \{1\}$ . The purpose of this paper is to give some results which show the role of  $Lf$ -functions in finding a fiber homotopically equivalent relation between two fibrations, over a common polyhedron base. Furthermore we will prove the equivalently between our results and Dold's theorem in fiber bundles, over a common suspension base of polyhedron spaces.

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## 1 Introduction

In what follows, for a topological space  $E$ ,  $\tilde{E}$  will denote a set of all constant path  $\tilde{e}$  into  $e \in E$ ,  $\bar{\alpha}$  the inverse path of  $\alpha \in E^I$ ,  $\star$  the usual path multiplication operation,  $\simeq$  the same homotopy type for spaces and homotopic for maps and  $\simeq_f$  a fiber homotopy.

First we recall from [9] that the *simplicial complex*  $K$  contains of a set  $\{v\}$  of vertices and a set  $\{s\}$  of finite nonempty subsets of  $\{v\}$  is called simplexes such that any set consisting of exactly one vertex in a simplex and any nonempty subset of simplex is a simplex.  $/K/$  denotes a set of all functions  $\beta$  from the set of vertices of  $K$  to  $I$  such that for any function  $\beta \in /K/$ , the set  $\{v \in K : \beta(v) = 0\}$  is a simplex and  $\sum_{v \in K} \beta(v) = 1$ . The topology on  $/K/$  is a topology which induced by metric  $d$  on  $/K/$  defined by  $d(\alpha, \beta) = \{\sum_{v \in K} [\alpha(v) - \beta(v)]^2\}^{0.5}$ . A topological space  $E$  is called a *polyhedron* if there is a simplicial complex  $K$  and a homeomorphism  $f : /K/ \longrightarrow E$ . A closed subspace  $A$  of  $E$  is called a *subpolyhedron* of  $E$  if there is simplicial complex  $L \subset K$  such that  $f(/L/) = A$ . And if  $A$  is a *subpolyhedron* of  $E$ , we say that the pair  $(E, A)$  *polyhedron pair*. In our paper, for any simplicial complex, any vertex belong to a finite simplexes, hence any polyhedron in this case will be an ANR (see [9]).

From [1] the *suspension*  $S(O)$  of  $O$  based a fixed point  $r_o \in O$  is defined to be the quotient space of  $O \times I$  in which for all  $b \in O$ ,  $(b, 0)$  is identified to  $(r_o, 0)$  and  $(b, 1)$  is identified to  $(r_o, 1)$ . Consider  $S(O)$  as the union of two cones, one of them is defined

$$S_0(O) = \{[(x, t)] \in S(O) : (x, t) \in O \times [0, 1/2]\}$$

with  $(x, 0)$  identified to  $(r_o, 0)$  and the other by

$$S_1(O) = \{[(x, t)] \in S(O) : (x, t) \in O \times [1/2, 1]\}$$

with  $(x, 1)$  identified to  $(r_o, 1)$ . The cones are always contractible spaces (see [7, 8]). If  $O$  is an absolute neighborhood retract (ANR) space, then  $S(O)$  is an ANR. Similarly, for polyhedron property, for more details see [5, 6].

Dold's theorem is one of the famous solutions for the classification problem in fiber bundles over the  $n$ -sphere bases  $S^n$ , (see[10]). James in [13] showed that Dold's theorem remains valid if we use suspensions of polyhedron space instead of  $n$ -spheres  $S^n$  for the base of bundles., that is, Dold's theorem will take the following form:

**Theorem 1.1.** [The Dold's theorem] Let  $\gamma = (E, f, S(O), F, G)$  and  $\gamma' = (E', f', S^n, F', G')$  be two fiber bundles over a suspension  $S(O)$  of polyhedron space  $O$  with locally compact fibers  $F$  and  $F'$ . Let  $\mu : (O, x_o) \rightarrow (G, e)$  and  $\mu' : (O, x_o) \rightarrow (G', e')$  be characteristic maps of  $\gamma$  and  $\gamma'$ , respectively and let  $i : G \rightarrow H(F, F)$  and  $i' : G' \rightarrow H(F', F')$  be the inclusion maps. Then  $\gamma$  and  $\gamma'$  are fiber homotopy equivalent if and only if there is homotopy equivalence  $g : F \rightarrow F'$  such that the maps

$$q(x) = g \circ (i \circ \mu)(x) \circ \overleftarrow{g} \quad \text{and} \quad q'(x) = (i' \circ \mu')(x)$$

from  $O$  into  $H(F', F')$  are homotopic, where  $H(F, F)$  is the set of all homotopy equivalences from  $F$  into  $F$  and  $\overleftarrow{g}$  is the inverse homotopy of  $g \in H(F, F)$ .

This paper is organized as follows. It consists of four sections. After this Introduction, Section 2 is devoted to some preliminaries. In Section 3 we shall start by giving some results about homotopy extension property and  $Lf$ -function properties. Next we show the role of  $Lf$ -function  $\Theta_{L_f}$  in finding fiber homotopy equivalence between two fibrations. Mainly, we prove the following theorem:

**Theorem 1.2.** *Let  $[E_1, f_1, O, F_{r_o}^1]$  and  $[E_2, f_2, O, F_{r_o}^2]$  be two fibrations over a polyhedron base  $O$ . Let  $O$  be the union of two subpolyhedra  $O_1$  and  $O_2$  such that  $O_1$  is a contractible in  $O$  to  $r_o \in O_3 = O_1 \cap O_2$  leaves  $r_o$  fixed and  $O_2$  is contractible to  $r_o$ . If  $O_3$  is subpolyhedra of  $O$ , then  $f_1$  and  $f_2$  are fiber homotopy equivalent if and only if they have conjugate  $Lf$ -functions by  $g \in H(F_{r_o}^1, F_{r_o}^2)$ .*

In section 4 we will apply  $Lf$ -function in fiber bundles by proving the equivalently between Theorem 1.2 and Dold's theorem.

All topological spaces in this paper will be assumed Hausdorff spaces.

## 2 Preliminaries

Recall [2] that the path space  $Pa(E, e_o) = \{\alpha \in E^I : \alpha(0) = e_o\}$  based at fixed point  $e_o$ , a loop space  $\Omega(E, e_o)$  and  $\tilde{E}$  are closed in a path space  $E^I$ . If  $E$  is a metrizable (*resp.* ANR), then  $E^I$ ,  $Pa(E, e_o)$  and  $\Omega(E, e_o)$  are metrizable (*resp.* ANR).

Let  $f : E \rightarrow O$  be a fibration with a base  $O$ , total space  $E$  and fiber space  $F_{r_o} = f^{-1}(r_o)$ , where  $r_o \in O$ . A map  $L_f : \Delta f \rightarrow E^I$  is called a *lifting function* for  $f$  if  $L_f(e, \alpha)(0) = e$  and  $f[L_f(e, \alpha)] = \alpha$  for all  $(e, \alpha) \in \Delta f$ , where  $\Delta f = \{(e, \alpha) \in E \times Pa(O) : f(e) = \alpha(0)\}$ . If  $L_f(e, f \circ \tilde{s}) = \tilde{e}$  for all  $e \in E$ , then the lifting function is called a *regular lifting function*. A fibration  $f$  is called *regular fibration* if it has regular lifting function (see [2]).

Curtis-Hurewicz theorem, [4], is one of the famous theorems in fibration theory which shows that any map is regular fibration if and only if it has regular lifting function.

**Theorem 2.1.** [Dold-Fadell Theorem] (see [3]) Let  $f_1 : E_1 \rightarrow O$  and  $f_2 : E_2 \rightarrow O$  be two fibrations over an ANR pathwise connected base  $O$ . Then  $f_1$  and  $f_2$  are fiber homotopy equivalent if and only if there is a fiber map  $h : E_1 \rightarrow E_2$  such that the restriction map of  $h$  on  $f_1^{-1}(r_o)$  is homotopy equivalence into  $f_2^{-1}(r_o)$ , for some  $r_o \in O$ .

A closed subspace  $A$  of a space  $E$  is said to have a *homotopy extension property* in  $E$  with respect to a space  $O$  if any map  $f : (E \times \{0\}) \cup (A \times I) \rightarrow O$  can be extended to a map  $F : E \times I \rightarrow O$ .

**Theorem 2.2.** (See[11]) For a polyhedron pair  $(E, A)$ ,  $A$  has a homotopy extension property in  $E$  with respect to any space  $O$ .

**Theorem 2.3.** (See [6]) Let  $A$  be a closed subspace of metrizable space  $E$  and  $O$  be an ANR space. Then  $A$  has a homotopy extension property in  $E$  with respect to a space  $O$ .

**Theorem 2.4.** (See [6]) An ANR closed subspace  $A$  of an ANR space  $E$  has a homotopy extension property in  $E$  with respect to any space  $O$ .

Recall [12] that if  $f : E \rightarrow O$  is a fibration and  $A$  is a subspace of  $O$  then the restriction map  $f|_{f^{-1}(A)} : f^{-1}(A) \rightarrow A$  of  $f$  is a fibration and we denote it by  $f|_A$ .

**Theorem 2.5.** (see [12]) Let  $f_1 : E_1 \rightarrow O$  and  $f_2 : E_2 \rightarrow O$  be two regular fibrations over a polyhedron base  $O$  and  $A$  be a subpolyhedron of  $O$ . Suppose that there are two fiber maps  $h_1, h_2 : f_1|_A \rightarrow f_2|_A$  such that  $h_1 \simeq_f h_2$ . Then if  $k_1$  has an extension fiber map  $H_1 : E_1 \rightarrow E_2$ ,  $h_2$  has an extension fiber map  $H_2 : E_1 \rightarrow E_2$  and  $H_1 \simeq_f H_2$ .

Here we recall the details of the definition of fiber bundle which will be used in the our results in Section 4.

**Definition 2.6.** [10] Let  $E, O$  and  $F$  be spaces. Let  $f : E \rightarrow O$  be a map of  $E$  onto  $O$  and  $G$  be group of all homeomorphisms of  $F$  onto  $F$  with as a binary usual composition operation  $\circ$ . Then  $\gamma = (E, f, O, F, G)$  is said to be a *fiber bundle over a base  $O$*  if there is an open covering  $\{V_j : j \in \Lambda\}$  of  $O$  (where  $\Lambda$  is an index set) and for each  $j \in \Lambda$ , there is a homeomorphism  $\theta_j : V_j \times F \rightarrow f^{-1}(V_j)$  such that:

1.  $f[\theta_j(b, y)] = b$  for all  $b \in V_j, y \in F$ .
2. For each pair  $i, j \in \Lambda$  and  $b \in V_i \cap V_j$ , the homeomorphism  $\theta_{jb}^{-1} \circ \theta_{ib} : F \rightarrow F$  corresponds to an element of  $G$ , where  $\theta_{kb} : F \rightarrow f^{-1}(b)$  defined by  $\theta_{kb}(y) = \theta_k(b, y)$  for all  $b \in V_k, y \in F, (k = i, j)$ .
3. For each pair  $i, j \in \Lambda$ , the function  $g_{ij} : V_i \cap V_j \rightarrow G$  given by  $g_{ij}(b) = \theta_{jb}^{-1} \circ \theta_{ib}$  is a map.

**Remark 2.7.** In fiber bundle  $\gamma = (E, f, O, F, G)$ , the maps  $\theta_j : V_j \times F \rightarrow f^{-1}(V_j)$  are called the *coordinate functions*, the maps  $g_{ij}(b) = \theta_{jb}^{-1} \circ \theta_{ib}$  are called *coordinate transformations*, the space  $E$  is called *bundle* over base  $O$ , and  $F$  is called *fiber of bundle  $E$* . We shall denote the identity element of a group  $G$  by  $\mathbf{g}$ , the inverse element  $g \in G$  by  $g^{-1}$ .

**Theorem 2.8.** [13] Let  $S^n$  be the  $n$ -sphere in  $R^{n+1}$ . For a fiber bundle  $\gamma = (E, f, S^n, F, G)$ , there is a characteristic map  $\mu : (S^{n-1}, x_o) \rightarrow (G, e)$ , where  $n > 0$  is a positive integer.

### 3 An $L_f$ -functions of fibration

In this section, we will define the  $L_f$ -function and study its properties. Next we will show its role in finding a fiber homotopically equivalent relation between two fibrations over a common polyhedron base.

**Theorem 3.1.** *Let  $f : E \longrightarrow O$  be a regular fibration. Let  $(X, A)$  be an ANR pair (resp. be Polyhedron pair). If there is a map  $G : (X \times \{0\}) \cup (A \times I) \longrightarrow E$  such that*

$$f[G(a, t)] = f[G(a, 0)] \quad \text{for } a \in A, t \in I,$$

*then there is a map  $H : X \times I \longrightarrow E$  such that  $H$  is an extension of  $G$ ,*

$$f[H(x, t)] = f[H(x, 0)] \quad \text{for } x \in X, t \in I.$$

**Proof.** By Theorem 2.4 if  $(X, A)$  is an ANR pair or by Theorem 2.2 if  $(X, A)$  is a Polyhedron pair we get that the map  $G$  can be extended to a map  $F : X \times I \longrightarrow E$ . For a path  $\alpha \in E^I$  and  $r \in I$ , we can define the path  $\alpha_r$  in  $E^I$  by  $\alpha_r(t) = \alpha[(1-t)r]$  for all  $t \in I$ . Hence we can define the map  $H : X \times I \longrightarrow E$  by

$$H(x, t) = L_f[F(x, t), f \circ F(x)_t](1) \quad \text{for } x \in X, t \in I.$$

At case  $X \times \{0\}$ , by the regularity of  $L_f$  and since  $F$  is an extension for  $G$ , we observe that for  $x \in X$ ,

$$\begin{aligned} H(x, 0) &= L_f(F(x, 0), f \circ F(x)_0)(1) \\ &= L_f(F(x, 0), f \circ \widetilde{F(x, 0)})(1) \\ &= [\widetilde{F(x, 0)}](1) = F(x, 0) = G(x, 0). \end{aligned}$$

At case  $A \times I$ , since for  $a \in A$  and  $r, t \in I$ ,

$$\begin{aligned} (f \circ F(a)_t)(r) &= f[F(a)((1-r)t)] = f[F(a, (1-r)t)] \\ &= f[G(a, (1-r)t)] = f[G(a, 0)] = f[G(a, t)] \\ &= f[F(a, t)] = [f \circ \widetilde{F(ax, t)}](r), \end{aligned}$$

then by the regularity of  $L_f$  we get that  $H(a, t) = F(a, t) = G(a, t)$  for all  $t \in I, a \in A$ . That is,  $H$  is an extension for  $G$ .

Finally, we also observe that

$$\begin{aligned} f[H(x, t)] &= f[L_f(F(x, t), f \circ F(x)_t)(1)] = (f \circ F(x)_t)(1) \\ &= f[F(x, 0)] = f[G(x, 0)] = f[H(x, 0)], \end{aligned}$$

for all  $x \in X, t \in I$ .  $\square$

We can give another rephrasing of Theorem above in the following corollary:

**Corollary 3.2.** *Let  $f_1 : E_1 \longrightarrow O$  and  $f_2 : E_2 \longrightarrow O$  be two regular fibrations. Let  $A$  be a closed subspace of  $O$  and  $(E_1, f_1^{-1}(A))$  be an ANR pair (resp. be Polyhedron pair). If there are two fiber maps  $k_1, k_2 : f_1|_A \longrightarrow f_2|_A$  such that  $k_1 \simeq_f k_2$  and  $k_1$  has an extension fiber map  $K_1 : E_1 \longrightarrow E_2$ , then  $k_2$  has an extension fiber map  $K_2 : E_1 \longrightarrow E_2$  and  $K_1 \simeq_f K_2$ .*

**Proof.** Since  $k_1 \simeq_f k_2$  then there is a homotopy  $R : f_1^{-1}(A) \times I \longrightarrow f_2^{-1}(A)$  between  $k_1 := R_0$  and  $k_2 := R_1$  such that  $f_2[R(a, t)] = f_1(a)$ . We can apply Theorem 3.1 on the regular fibration  $f_2$  by taking  $(X, A) := (E_1, f_1^{-1}(A))$  and

$$G(e, t) = \begin{cases} R(e, t) & \text{for } (e, t) \in f_1^{-1}(A) \times I \\ K_1(e) & \text{for } (e, t) \in E_1 \times \{0\}, \end{cases}$$

to get the extension homotopy  $H$  of  $G$ . Hence  $H_0 = K_1$  and take  $K_2 = H_1$ . Then  $H$  is a homotopy between two fiber maps  $K_1$  and  $K_2$  and we get

$$f_2[H(e, t)] = f_2[H(e, 0)] = f_2[K_1(e)] = f_1(e)$$

for all  $e \in E_1, t \in I$ . That is,  $K_1 \simeq_f K_2$   $\square$

**Remark 3.3.** For an ANR spaces  $E$  and  $E_2$ , Theorem 3.1 and Corollary 3.2 will remain valid if we put  $A$  a closed subspace of metrizable space  $X$  instead of an ANR pair  $(X, A)$  since we can use Theorem 2.3 in the proofs. For a polyhedron base  $O$ , Corollary 3.2 will lead us to Theorem 2.5.

**Definition 3.4.** Let  $f : E \longrightarrow O$  be a fibration with fiber space  $F_{r_o} = f^{-1}(r_o)$ , where  $r_o \in O$ . By the  $Lf$ -function for fibration  $f$  induced by a lifting function  $L_f$  we mean a map  $\Theta_{L_f} : \Omega(O, r_o) \times F_{r_o} \longrightarrow F_{r_o}$  which is defined by

$$\Theta_{L_f}(\alpha, e) = L_f(e, \alpha)(1) \quad \text{for } e \in F_{r_o}, \alpha \in \Omega(O, r_o).$$

Henceforth, we will denote by  $[E, f, O, F_{r_o}]$  the regular fibration  $f : E \longrightarrow O$  with an  $Lf$ -function  $\Theta_{L_f} : \Omega(O, r_o) \times F_{r_o} \longrightarrow F_{r_o}$ , induced by the lifting function  $L_f$  and with a fiber space  $F_{r_o} = f^{-1}(r_o)$ , where  $r_o \in O$ .

**Theorem 3.5.** Let  $[E, f, O, F_{r_o}]$  be a fibration with metrizable spaces  $E$  and  $O$ . Let  $\Theta : \Omega(O, r_o) \times F_{r_o} \longrightarrow F_{r_o}$  be a map such that  $\Theta_{L_f} \simeq \Theta$  and  $\Theta(\tilde{r}_o, e) = e$  for all  $e \in F_{r_o}$ . If  $E$  is an ANR, then there is a regular lifting function  $L'_f$  for  $f$  which induces  $\Theta$ . That is,  $\Theta$  is the  $Lf$ -function for  $f$ .

**Proof.** Firstly, by the hypothesis,  $\Theta_{L_f} \simeq \Theta$ . Then there is a homotopy

$$R : [\Omega(O, r_o) \times F_{r_o}] \times I \longrightarrow F_{r_o}$$

such that  $R[(\alpha, e), 0] = \Theta_{L_f}(\alpha, e)$  and  $R[(\alpha, e), 1] = \Theta(\alpha, e)$  for all  $e \in F_{r_o}, \alpha \in \Omega(O, r_o)$ . We observe that  $\Theta_{L_f}$  is extendable to a map  $\Theta'_{L_f} : Pa(O, r_o) \times F_{r_o} \longrightarrow E$  defined by

$$\Theta'_{L_f}(\alpha, e) = L_f(e, \alpha)(1) \quad \text{for } e \in F_{r_o}, \alpha \in Pa(O, r_o)$$

having the property

$$f[\Theta'_{L_f}(\alpha, e)] = \alpha(1) \quad \text{for } e \in F_{r_o}, \alpha \in Pa(O, r_o).$$

This implies that  $R$  and  $\Theta'_{L_f}$  give us a map from

$$[Pa(O, r_o) \times F_{r_o} \times \{0\}] \cup [\Omega(O, r_o) \times F_{r_o} \times I]$$

in to  $E$ . Since  $\Omega(O, r_o) \times F_{r_o}$  is a closed in a metrizable space  $Pa(O, r_o) \times F_{r_o}$  then by Corollary 3.2 and Remark 3.3,  $\Theta$  can be extended to a map  $\Theta' : Pa(O, r_o) \times F_{r_o} \longrightarrow E$  such that

$$f[\Theta'(\alpha, e)] = \alpha(1) \quad \text{for } e \in F_{r_o}, \alpha \in \Omega(O, r_o).$$

Secondly, for  $\alpha \in O^I$  and  $r \in I$ , define two paths  $\alpha_r, \alpha^r \in O^I$  by

$$\alpha_r(t) = \alpha(rt) \quad \text{and} \quad \alpha^r(t) = \alpha(r + (1-r)t) \quad \text{for } t \in I.$$

Hence define a homotopy  $H' : [\Omega(O, r_o) \times F_{r_o}] \times I \longrightarrow F_{r_o}$  by

$$H'[(\alpha, e), t] = L_f[\Theta'(\alpha_t, e), \alpha^t](1) \quad \text{for } t \in I, e \in F_{r_o}, \alpha \in \Omega(O, r_o).$$

Hence by the hypothesis and the regularity for  $L_f$ , we observe that

$$\begin{aligned} H'[(\alpha, e), 0] &= L_f[\Theta'(\alpha_0, e), \alpha^0](1) \\ &= L_f[\Theta'(\tilde{r}_o, e), \alpha](1) \\ &= L_f(e, \alpha)(1) = \Theta_{L_f}(\alpha, e), \\ H'[(\alpha, e), 1] &= L_f[\Theta'(\alpha_1, e), \alpha^1](1) \\ &= L_f[\Theta'(\alpha, e), \tilde{r}_o](1) \\ &= \Theta'(\alpha, e) = \Theta(\alpha, e), \end{aligned}$$

for all  $e \in F_{r_o}, \alpha \in \Omega(O, r_o)$  and

$$\begin{aligned} H'[(\tilde{r}_o, e), t] &= L_f[\Theta'((\tilde{r}_o)_t, e), (\tilde{r}_o)^t](1) \\ &= L_f[\Theta'(\tilde{r}_o, e), \tilde{r}_o](1) \\ &= L_f(e, \tilde{r}_o)(1) = e, \end{aligned}$$

for all  $e \in F_{r_o}$ . Again we can apply Theorem 3.1 and Remark 3.3 by taking

$$A := B \cup C \quad \text{and} \quad X := \Delta' f = \{(\alpha, e) \in Pa(O) \times E : \alpha(0) = f(e)\}$$

where  $B = [\Omega(O, r_o) \times F_{r_o}]$ , and  $C = [(\tilde{O} \times E) \cap \Delta' f]$ , and a map  $G : (X \times \{0\}) \cup (A \times I) \longrightarrow E$  given by

$$G[(\alpha, e), t] = \begin{cases} H'[(\alpha, e), t] & \text{for } [(\alpha, e), t] \in B \times I, \\ e & \text{for } [(\alpha, e), t] \in C \times I, \\ L_f(e, \alpha)(1) & \text{for } [(\alpha, e), t] \in \Delta' f \times \{0\}, \end{cases}$$

Hence there is a map  $H : \Delta' f \times I \longrightarrow E$  such that  $H$  is an extension of  $G$  and

$$f\{H[(\alpha, e), t]\} = f\{H[(\alpha, e), 0]\} \quad \text{for } (\alpha, e) \in \Delta' f, t \in I.$$

Finally, we can define a map  $L'_f : \Delta f \longrightarrow E^I$  by

$$L'_f(e, \alpha)(t) = H[(\alpha_t, e), 1] \quad \text{for } (e, \alpha) \in \Delta f, t \in I.$$

Now we will show that  $L'_f$  is a regular lifting function for  $f$  as follows:

1. For  $(e, \alpha) \in \Delta f$ , we have that

$$L'_f(e, \alpha)(0) = H[(\alpha_0, e), 1] = G[(\alpha(0), e), 1] = e;$$

2. For  $(e, \alpha) \in \Delta f$  and  $t \in I$ , we have that

$$f[L'_f(e, \alpha)(t)] = f\{H[(\alpha_t, e), 1]\} = f\{H[(\alpha_t, e), 0]\} = f[L_f(e, \alpha_t)(1)] = \alpha(t);$$

3. For  $e \in S$ ,

$$L'_f(e, f \circ \widetilde{e})(t) = H[(\widetilde{f(e)})_t, e], 1] = G[(\widetilde{f(e)})_t, e], 1] = e.$$

Hence  $L'_f$  is a regular lifting function of  $f$  and for  $(\alpha, e) \in \Omega(O, r_o) \times F_{r_o}$ ,

$$L'_f(e, \alpha)(1) = H[(\alpha_1, e), 1] = H[(\alpha, e), 1] = H'[(\alpha, e), 1] = \Theta(\alpha, e).$$

That is,  $\Theta$  is an  $Lf$ -function for  $f$  induced by the regular lifting function  $L'_f$ .  $\square$

It is clear that the lifting function for any fibration no need to be unique and the definition of the  $Lf$ -function depends on the lifting function. So the  $Lf$ -function no need to be unique but it is uniquely determined up to a homotopy class as it is shown in the following theorem.

**Theorem 3.6.** *Let  $[E, f, O, F_{r_o}]$  be a fibration. If  $f$  has two lifting functions  $L_f$  and  $L'_f$ , then the  $Lf$ -functions  $\Theta_{L_f}$  and  $\Theta_{L'_f}$  are homotopic.*

**Proof.** For  $\alpha \in O^I$  and  $r \in I$ , we can define two paths  $\alpha_r$  and  $\alpha^r$  in  $O^I$  by

$$\alpha_r(t) = \alpha(rt) \quad \text{and} \quad \alpha^r(t) = \alpha(r + (1-r)t) \quad \text{for } t \in I.$$

Hence we can define a homotopy

$$H : [\Omega(O, r_o) \times F_{r_o}] \times I \longrightarrow F_{r_o}$$

by

$$H[(\alpha, e), t] = L_f[L'_f(e, \alpha_t)(1), \alpha^t](1) \quad \text{for } t \in I, e \in F_{r_o}, \alpha \in \Omega(O, r_o).$$

By the regularity for  $L_f$  and  $L'_f$ , we get that

$$\begin{aligned} H[(\alpha, e), 0] &= L_f[L'_f(e, \alpha_0)(1), \alpha^0](1) = L_f[L'_f(e, \widetilde{r_o})(1), \alpha](1) \\ &= L_f(e, \alpha)(1) \\ &= \Theta_{L_f}(\alpha, e), \end{aligned}$$

and

$$\begin{aligned} H[(\alpha, e), 1] &= L_f[L'_f(e, \alpha_1)(1), \alpha^1](1) = L_f[L'_f(e, \alpha)(1), \widetilde{r_o}](1) \\ &= L'_f(e, \alpha)(1) \\ &= \Theta_{L'_f}(\alpha, e), \end{aligned}$$

for all  $e \in F_{r_o}$ ,  $\alpha \in \Omega(O, r_o)$ . Hence  $\Theta_{L_f}$  and  $\Theta_{L'_f}$  are homotopic.  $\square$

**Definition 3.7.** Let  $[E_1, f_1, O, F_{r_o}^1]$  and  $[E_2, f_2, O, F_{r_o}^2]$  be two fibrations. The  $Lf$ -functions  $\Theta_{L_{f_1}}$  and  $\Theta_{L_{f_2}}$  are said to be *conjugate* if there is  $g \in H(F_{r_o}^1, F_{r_o}^2)$  such that

$$\Theta_{L_{f_1}} \simeq \overleftarrow{g} \circ \Theta_{L_{f_2}} \circ (i_{\Omega(O, r_o)} \times g),$$

where  $i_{\Omega(O, r_o)}$  dentes the identity map of  $\Omega(O, r_o)$  onto itself.

**Theorem 3.8.** *Let  $[E_1, f_1, O, F_{r_o}^1]$  and  $[E_2, f_2, O, F_{r_o}^2]$  be two fibrations. If  $f_1$  and  $f_2$  are fiber homotopic equivalent, then  $\Theta_{L_{f_1}}$  and  $\Theta_{L_{f_2}}$  are conjugate  $Lf$ -functions.*

**Proof.** Let  $f_1$  and  $f_2$  are fiber homotopic equivalent by two fiber maps

$$h : E_1 \longrightarrow E_2 \quad \text{and} \quad g : E_2 \longrightarrow E_1.$$

For  $\alpha \in O^I$  and  $r \in I$ , we can define two paths  $\alpha_r$  and  $\alpha'_r$  by

$$\alpha_r(t) = \alpha(rt) \quad \text{and} \quad \alpha'_r(t) = \alpha[r + (1-r)t] \quad \text{for } t \in I.$$

Hence we can define a homotopy  $H : \Delta f_1 \times I \longrightarrow E_1$  by

$$H[(\alpha, e), t] = g\{L_{f_2}\{h[L_{f_1}(e, \alpha_t)(1)], \alpha'_t\}(1)\} \quad \text{for } t \in I, (e, \alpha) \in \Delta f_1$$

By the regularity for  $L_{f_1}$  and  $L_{f_2}$ , we get that

$$\begin{aligned} H[(\alpha, e), 0] &= g\{L_{f_2}\{h[L_{f_1}(e, \alpha_0)(1)], \alpha'_0\}(1)\} \\ &= g\{L_{f_2}\{h[L_{f_1}(e, f_1 \circ \tilde{e})(1)], \alpha\}(1)\} \\ &= g[L_{f_2}(h(e), \alpha)(1)], \end{aligned}$$

and

$$\begin{aligned} H[(\alpha, e), 1] &= g\{L_{f_2}\{h[L_{f_1}(e, \alpha_1)(1)], \alpha'_1\}(1)\} \\ &= g\{L_{f_2}\{h[L_{f_1}(e, \alpha)(1)], \widetilde{\alpha(1)}\}(1)\} \\ &= g\{L_{f_2}\{h[L_{f_1}(e, \alpha)(1)], f_1[L_{f_1}(\widetilde{e, \alpha})(1)]\}(1)\} \\ &= g\{L_{f_2}\{h[L_{f_1}(e, \alpha)(1)], (f_2 \circ h)[L_{f_1}(\widetilde{e, \alpha})(1)]\}(1)\} \\ &= g\{L_{f_2}\{h[L_{f_1}(e, \alpha)(1)], f_2[h(L_{f_1}(\widetilde{e, \alpha})(1))]\}(1)\} \\ &= g\{h[L_{f_1}(e, \alpha)(1)]\} \\ &= (g \circ h)[L_{f_1}(e, \alpha)(1)], \end{aligned}$$

for all  $(e, \alpha) \in \Delta f_1$ . Consider the composition

$$\Omega(O, r_o) \times F_{r_o}^1 \xrightarrow{i_{\Omega(O, r_o) \times h_o}} \Omega(O, r_o) \times F_{r_o}^2 \xrightarrow{\Theta_{L_{f_2}}} F_{r_o}^2 \xrightarrow{g_o} F_{r_o}^1$$

where  $h_o = h|_{F_{r_o}^1}$  and  $g_o = g|_{F_{r_o}^2}$ . Hence define a homotopy  $G : \Omega(O, r_o) \times F_{r_o}^1 \times I \longrightarrow F_{r_o}^1$  as a restriction map  $G = H|_{\Omega(O, r_o) \times F_{r_o}^1}$  of  $H$  on  $\Omega(O, r_o) \times F_{r_o}^1$ . Then we get that

$$\begin{aligned} G[(\alpha, e), 0] &= g_o[L_{f_2}(h_o(e), \alpha)(1)] \\ &= g_o[\Theta_{L_{f_2}}(\alpha, h_o(e))] \\ &= [g_o \circ \Theta_{L_{f_2}} \circ (i_{\Omega(O, r_o)} \times h_o)](\alpha, e), \end{aligned}$$

and

$$\begin{aligned} G[(\alpha, e), 1] &= (g_o \circ h_o)[L_{f_1}(e, \alpha)(1)] \\ &= (g_o \circ h_o)[\Theta_{L_{f_1}}(\alpha, e)] \\ &= [(g_o \circ h_o) \circ \Theta_{L_{f_1}}](\alpha, e), \end{aligned}$$



for all  $e \in F_{r_o}^1$  and  $\alpha \in \Omega(O, r_o)$ . Hence

$$g_o \circ \Theta_{L_{f_2}} \circ (id_{\Omega(O, r_o)} \times h_o) \simeq (g_o \circ h_o) \circ \Theta_{L_{f_1}}.$$

Since  $g_o \circ h_o \simeq id_{F_{r_o}^1}$ , then  $\overleftarrow{h_o} \circ \Theta_{L_{f_2}} \circ (id_{\Omega(O, r_o)} \times h_o) \simeq \Theta_{L_{f_1}}$ . Hence  $\Theta_{L_{f_1}}$  and  $\Theta_{L_{f_2}}$  are conjugate  $Lf$ -functions.  $\square$

**Lemma 3.9.** *Let  $[E, f, O, F_{r_o}]$  be a fibration. Then the maps  $D, D_o : \Delta f \longrightarrow E$  defined by*

$$D(e, \alpha) = L_f[L_f(e, \alpha)(1), \overline{\alpha}](1) \quad \text{and} \quad D_o(e, \alpha) = e,$$

for all  $(e, \alpha) \in \Delta f$ , are homotopic.

**Proof.** For  $\alpha \in O^I$  and  $r \in I$ , define paths  $\alpha_r, \alpha'_r$  and  $\alpha''_r$  in  $O$  by

$$\alpha_r(t) = \alpha(rt), \quad \alpha'_r(t) = \alpha[r + (1-r)t] \quad \text{and} \quad \alpha''_r(t) = \alpha[2r(1-t)],$$

for all  $t \in I$ . Define two homotopies  $H : \Delta f \times I \longrightarrow E$  by

$$H[(e, \alpha), t] = L_f[L_f(e, \alpha_t)(1), \alpha'_t](1) \quad \text{for } t \in I, (e, \alpha) \in \Delta f,$$

and a homotopy  $G : O^I \times I \longrightarrow O^I$  by

$$[G(\alpha, r)](t) = \begin{cases} \alpha_r(t) & \text{for } 0 \leq t \leq 1/2, \\ \alpha''_r(t) & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

for all  $\alpha \in O^I, r \in I$ . Hence define a homotopy  $F : \Delta f \times I \longrightarrow E$  by

$$F[(e, \alpha), t] = H[(e, G(\alpha, t)), 1/2] \quad \text{for } t \in I, (e, \alpha) \in \Delta f.$$

By the regularity for  $L_f$  we observe that for  $(e, \alpha) \in \Delta f$ ,

$$\begin{aligned} F[(e, \alpha), 1] &= H[(e, G(\alpha, 1)), 1/2] &= H[(e, \alpha \star \overline{\alpha}), 1/2] \\ &= L_f\{K[e, (\alpha \star \overline{\alpha})_{1/2}], (\alpha \star \overline{\alpha})'_{1/2}\}(1) \\ &= L_f[K(e, \alpha), \overline{\alpha}](1) \\ &= L_f[L_f(e, \alpha)(1), \overline{\alpha}](1) \\ &= D(e, \alpha) \end{aligned}$$

for all  $(e, \alpha) \in \Delta f$ , and

$$\begin{aligned} F[(e, \alpha), 0] &= H[(e, G(\alpha)), 0](1/2) &= H[(e, \widetilde{\alpha(0)}), 1/2] \\ &= L_f\{K[e, \widetilde{\alpha(0)}_{1/2}], \widetilde{\alpha(0)}'_{1/2}\}(1) \\ &= L_f\{K[e, \widetilde{\alpha(0)}], \widetilde{\alpha(0)}\}(1) \\ &= L_f[L_f(e, f \circ \widetilde{e})(1), f \circ \widetilde{e}](1) \\ &= L_f(e, f \circ \widetilde{e})(1) \\ &= e = D_o(e, \alpha) \end{aligned}$$

for all  $(e, \alpha) \in \Delta f$ . Hence  $D$  and  $D_o$  are homotopic.  $\square$

In the proof of Lemma above we get that the homotopy  $F$  has the following property:

$$f\{F[(e, \alpha), t]\} = \alpha(0) \quad \text{for } (e, \alpha) \in \Delta f. \tag{1}$$

**Remark 3.10.** In Lemma 3.9, for  $t \in I$ , we can define a map  $\Theta_{L_f}^t : \Omega(O, r_o) \times F_{r_o} \longrightarrow F_{r_o}$  by

$$\Theta_{L_f}^t(\alpha, s) = H(s, \alpha)(t) \quad \text{for } \alpha \in \Omega(O, r_o), s \in F_{r_o}.$$

By the regularity for  $L_f$ , we observe that

$$\Theta_{L_f}^0(\alpha, s) = \Theta_{L_f}^1(\alpha, s) = \Theta_{L_f}(\alpha, s).$$

By the definition of  $\Theta_{L_f}^t$  we get that for  $t \in I$ ,  $\Theta_{L_f}^t \simeq \Theta_{L_f}$ . Also we observe that  $\Theta_{L_f}^t(\tilde{r}_o, s) = s$  for all  $s \in F_{r_o}$ .

**Proof of Theorem (1.2).** *Necessity:* If  $f_1$  and  $f_2$  are fiber homotopy equivalent then by Theorem 3.8 they have conjugate  $L_f$ -functions.

*Sufficiency:* Since  $O_1$  and  $O_2$  are contractible to  $r_o \in O_3$ , then there are two homotopy maps  $R_1 : O_1 \times I \longrightarrow O$  and  $R_2 : O_2 \times I \longrightarrow O$  such that

$$R_1(x, 0) = x \quad R_1(x, 1) = r_o \quad \text{for } x \in O_1,$$

and

$$R_2(x, 0) = x \quad R_2(x, 1) = r_o \quad \text{for } x \in O_2,$$

respectively. We will denote the path:  $t \longrightarrow R_1(x, t)$  by  $R_1^x$  for  $x \in O_1$  and the path:  $t \longrightarrow R_2(x, t)$  by  $R_2^x$  for  $x \in O_2$ .

In Figure 1, let  $f_{ij} = f_i|_{O_j}$ , where  $i = 1, 2$  and  $j = 1, 2, 3$ . Now we can define a map  $h_1 : f_1^{-1}(O_1) \longrightarrow f_2^{-1}(O_1)$  by

$$h_1(e) = L_{f_2}\{g[L_{f_1}(e, R_1^{f_1(e)})(1)], \overline{R_1^{f_1(e)}}\}(1) \quad \text{for } e \in f_1^{-1}(O_1),$$

and a map  $h_2 : f_1^{-1}(O_2) \longrightarrow f_2^{-1}(O_2)$  by

$$h_2(e) = L_{f_2}\{g[L_{f_1}(e, R_2^{f_1(e)})(1)], \overline{R_2^{f_1(e)}}\}(1) \quad \text{for } e \in f_1^{-1}(O_2).$$

It is clear that

$$(f_2 \circ h_1)(e) = \overline{R_1^{f_1(e)}}(1) = R_1^{f_1(e)}(0) = f_1(e)$$

for all  $e \in f_1^{-1}(O_1)$  and

$$(f_2 \circ h_2)(e) = \overline{R_2^{f_1(e)}}(1) = R_2^{f_1(e)}(0) = f_1(e)$$

for all  $e \in f_1^{-1}(O_2)$ . That is,  $h_1$  and  $h_2$  are fiber maps.

Also in Figure 1, we can define a map  $k_1 : f_1^{-1}(O_3) \longrightarrow f_1^{-1}(O_3)$  by

$$k_1(e) = L_{f_1}[L_{f_1}(e, R_2^{f_1(e)})(1), \overline{R_2^{f_1(e)}}](1) \quad \text{for } e \in f_1^{-1}(O_3),$$

and a map  $k_2 : f_2^{-1}(O_3) \longrightarrow f_2^{-1}(O_3)$  by

$$k_2(e) = L_{f_2}[L_{f_2}(e, R_1^{f_2(e)})(1), \overline{R_1^{f_2(e)}}](1) \quad \text{for } e \in f_2^{-1}(O_3),$$

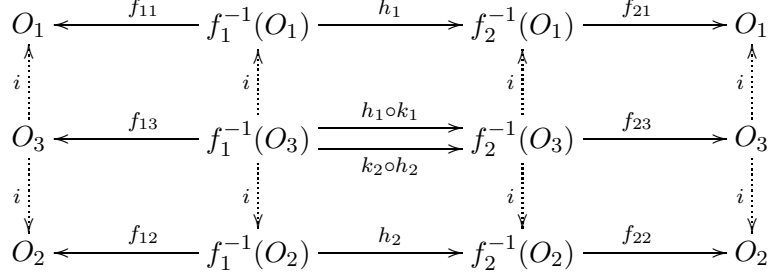


Figure 1

respectively. It is clear that

$$(f_1 \circ k_1)(e) = \overline{R_2^{f_1(e)}}(1) = R_2^{f_1(e)}(0) = f_1(e)$$

for all  $e \in f_1^{-1}(O_3)$  and

$$(f_2 \circ k_2)(e) = \overline{R_1^{f_2(e)}}(1) = R_1^{f_2(e)}(0) = f_2(e)$$

for all  $e \in f_2^{-1}(O_3)$ . That is,  $k_1$  and  $k_2$  are fiber maps. By Lemma 3.9, we get that

$$k_1 \simeq_f i_{f_1^{-1}(O_3)} \quad \text{and} \quad k_2 \simeq_f i_{f_2^{-1}(O_3)}.$$

Then

$$h_1 \circ k_1 \simeq_f h_1 \quad \text{and} \quad k_2 \circ h_2 \simeq_f h_2.$$

That is, to show that  $h_1 \simeq_f h_2$  as fiber maps of  $f_1^{-1}(O_3)$  into  $f_2^{-1}(O_3)$ , it is sufficient to show that  $h_1 \circ k_1 \stackrel{f}{\simeq} k_2 \circ h_2$  as fiber maps of  $f_1^{-1}(O_3)$  into  $f_2^{-1}(O_3)$ . By Remark 3.10, we get that

$$\begin{aligned} (h_1 \circ k_1)(e) &= L_{f_2} \{ g \{ \Theta_{L_{f_1}}^{1/2} [\overline{R_2^{f_1(e)}} \star R_1^{f_1(e)}, L_{f_1}(e, R_2^{f_1(e)})(1)] \}, \overline{R_1^{f_1(e)}} \} (1) \\ &= L_{f_2} \{ (g \circ \Theta_{L_{f_1}}^{1/2}) [\overline{R_2^{f_1(e)}} \star R_1^{f_1(e)}, L_{f_1}(e, R_2^{f_1(e)})(1)], \overline{R_1^{f_1(e)}} \} (1), \end{aligned}$$

and

$$\begin{aligned} (k_2 \circ h_2)(e) &= L_{f_2} \{ \Theta_{L_{f_2}}^{1/2} \{ \overline{R_2^{f_1(e)}} \star R_1^{f_1(e)}, g[L_{f_1}(e, R_2^{f_1(e)})(1)] \}, \overline{R_1^{f_1(e)}} \} (1) \\ &= L_{f_2} \{ [\Theta_{L_{f_2}}^{1/2} \circ (1 \times g)] \{ \overline{R_2^{f_1(e)}} \star R_1^{f_1(e)}, L_{f_1}(e, R_2^{f_1(e)})(1) \}, \overline{R_1^{f_1(e)}} \} (1), \end{aligned}$$

for all  $e \in f_1^{-1}(O_3)$ , where  $1 = i_{\Omega(O, r_o)}$ .

By the hypothesis  $\Theta_{L_{f_1}} \simeq \overleftarrow{g} \circ \Theta_{L_{f_2}} \circ (i_{\Omega(O, r_o)} \times g)$ , e.g.,

$$g \circ \Theta_{L_{f_1}} \simeq \Theta_{L_{f_2}} \circ (i_{\Omega(O, r_o)} \times g),$$

and by Remark 3.10, we get  $\Theta_{L_{f_1}} \simeq \Theta_{L_{f_1}}^{1/2}$  and  $\Theta_{L_{f_2}} \simeq \Theta_{L_{f_2}}^{1/2}$ . That is,

$$g \circ \Theta_{L_{f_1}}^{1/2} \simeq \Theta_{L_{f_2}}^{1/2} \circ (i_{\Omega(O, r_o)} \times g).$$

Hence by Lemma 3.9 again, we get

$$h_1 \circ k_1 \simeq_f k_2 \circ h_2 \implies h_1 \simeq_f h_2, \quad (2)$$

as fiber maps of  $f_1^{-1}(O_3)$  into  $f_2^{-1}(O_3)$ .

Now in two fibrations  $f_{13} = f_1|_{O_3}$  and  $f_{23} = f_2|_{O_3}$ , since  $O_3$  is a subpolyhedra of  $O_2$ ,  $h_1 \simeq_f h_2$  as maps of  $f_1^{-1}(O_3)$  into  $f_2^{-1}(O_3)$  and  $h_2$  is defined on  $f_1^{-1}(O_2)$ , then by Theorem 2.5  $h_1$  can be extended as fiber map to all of  $f_1^{-1}(O_2)$ . Since  $h_1$  is defined on  $f_1^{-1}(O_1)$  then  $h_1$  gives a fiber map  $h$  of  $E_1$  into  $E_2$ . Since  $O_1$  is contractible to  $r_o$  leaves  $r_o$  a fixed, then  $R_1^{r_o} = \tilde{r}_o$ . Hence for  $e \in F_{r_o}^1 \subseteq f_1^{-1}(O_3)$ ,

$$\begin{aligned} h(e) = h_1(e) &= L_{f_2}\{g[L_{f_1}(e, R_1^{f_1(e)})(1)], \overline{R_1^{f_1(e)}}\}(1) \\ &= L_{f_2}\{g[L_{f_1}(e, R_1^{r_o})(1)], \overline{R_1^{r_o}}\}(1) \\ &= L_{f_2}\{g[L_{f_1}(e, \tilde{r}_o)(1)], \tilde{r}_o\}(1) \\ &= L_{f_2}(g(e), \tilde{r}_o)(1) = g(e). \end{aligned}$$

That is,  $h$  as map:  $F_{r_o}^1 \longrightarrow F_{r_o}^2$  is a homotopy equivalent. Since  $O$  is an ANR and it is clear that  $O$  is a pathwise connected ( $O$  is the union for two contractible spaces and  $O_3 \neq \emptyset$ ), then by Fadell-Dold theorem,  $f_1$  and  $f_2$  are fiber homotopy equivalent.  $\square$

## 4 Applying $Lf$ -function in fiber bundles

Here we apply the  $Lf$ -function in fiber bundles by proving the equivalently between Theorem 1.2 and Dold's theorem.

Firstly, we will give some propositions which help us to make comparing between Dold's theorem and Theorem 1.2.

In the following proposition we will prove the converse of Theorem 2.8.

**Proposition 4.1.** Let  $G$  be a group of all homeomorphisms of space  $F$  with as binary usual composition operation  $\circ$ . If there is a map  $\mu : (S^{n-1}, x_o) \longrightarrow (G, \mathbf{g})$ , then there is bundle  $E$  over sphere  $S^n$  and a map  $f : E \longrightarrow S^n$  such that  $\gamma = (E, f, S^n, F, G)$  is fiber bundle.

**Proof.** Let  $S^n = V_1 \cup V_2$ , where each  $V_i$  is an open  $n$ -cell such that  $V_1 \cap V_2$  is a trip a round  $S^{n-1}$  and there is a retraction  $r : V_1 \cap V_2 \longrightarrow S^{n-1}$  ( see[13]). Now define maps

$$g_{ii} : V_i \longrightarrow G, \quad g_{ii}(x) = \mathbf{g} \quad \forall x \in V_i, (i = 1, 2),$$

$$g_{12} : V_1 \cap V_2 \longrightarrow G, \quad g_{12}(x) = (\mu \circ r)(x) \quad \forall x \in V_1 \cap V_2,$$

and

$$g_{21} : V_1 \cap V_2 \longrightarrow G, \quad g_{21}(x) = [g_{12}(x)]^{-1} \quad \forall x \in V_1 \cap V_2.$$

Let  $J = \{1, 2\}$  be a space with the discrete topology and  $T \subset S^n \times F \times J$  be the set defined by

$$T = \{(x, y, j) : x \in V_j, y \in F, j \in J\}.$$

Define an equivalent relation  $\equiv$  on  $T$  by

$$(x_1, y_1, j) \equiv (x_2, y_2, k) \iff x_1 = x_2 \quad \text{and} \quad g_{kj}(x_1)(y_1) = y_2,$$

where  $(x_1, y_1, j), (x_2, y_2, k) \in T$ . Then put  $E$  to be the space of equivalence classes obtained with the quotient topology. Hence define a map  $f : E \longrightarrow S^n$  by

$$f([(x, y, j)]) = x \quad \forall [(x, y, j)] \in E,$$

and the maps  $\theta_j : V_j \times F \longrightarrow f^{-1}(V_j)$  defined by

$$\theta_j(x, y) = [(x, y, j)] \quad \forall (x, y) \in V_j \times F.$$

Hence it is clear that  $\gamma = (E, f, S^n, F, G)$  is a fiber bundle.  $\square$

**Proposition 4.2.** Let  $[E, f, O, F_{r_o}]$  be a fibration with locally compact fiber space  $F_{r_o}$ . Then the function  $\phi : \Omega(O, r_o) \longrightarrow F_{r_o}^{F_{r_o}}$  given by

$$\phi(w)(e) = \Theta_{L_f}(w, e) \quad \text{for } w \in \Omega(O, r_o), e \in F_{r_o},$$

is a map from  $\Omega(O, r_o)$  into  $H(F_{r_o}, F_{r_o})$ .

**Proof.** Since a Hausdorff space  $F_{r_o}$  is a locally compact then  $\phi$  is continuous function (see [2] Proposition A.14 P.530). Now we will prove that for  $w \in \Omega(O, r_o)$ ,  $\phi(w)$  is homotopy equivalence from  $F_{r_o}$  into  $F_{r_o}$ . For  $w \in \Omega(O, r_o)$ , we can define a map  $\overleftarrow{\phi}(w) : F_{r_o} \longrightarrow F_{r_o}$  by

$$\overleftarrow{\phi}(w)(e) = \Theta_{L_f}(\overline{w}, e) \quad \text{for } e \in F_{r_o}.$$

Then we get that

$$[\phi(w) \circ \overleftarrow{\phi}(w)](e) = L_f[L_f(e, \overline{w})(1), w](1) \quad \text{for } e \in F_{r_o},$$

and

$$\overleftarrow{\phi}(w) \circ \phi(w)(e) = L_f[L_f(e, w)(1), \overline{w}](1) \quad \text{for } e \in F_{r_o}.$$

Then by Lemma 3.9,

$$\phi(w) \circ \overleftarrow{\phi}(w) \simeq i_{F_{r_o}} \quad \text{and} \quad \overleftarrow{\phi}(w) \circ \phi(w) \simeq i_{F_{r_o}}.$$

Hence  $\phi(w) \in H(F_{r_o}, F_{r_o})$ . Therefore  $\phi$  is a map from  $\Omega(O, r_o)$  into  $H(F_{r_o}, F_{r_o})$ .  $\square$

**Proposition 4.3.** Let  $\gamma = [E, f, O, F_{r_o}, G]$  be a fiber bundle and fibration with locally compact fiber  $F_{r_o}$ . Then the function  $\phi : \Omega(O, r_o) \longrightarrow F_{r_o}^{F_{r_o}}$  given by

$$\phi(w)(e) = \Theta_{L_f}(w, e) \quad \text{for } w \in \Omega(O, r_o), e \in F_{r_o},$$

is a map from  $\Omega(O, r_o)$  into  $G$ .

**Proof.** Since  $F$  is a locally compact, then  $\phi$  is continuous function. For  $w \in O^I$ , let  $F_{w(0)} = f^{-1}(w(0))$  and  $F_{w(1)} = f^{-1}(w(1))$ . Then the map  $h : F_{w(0)} \longrightarrow F_{w(1)}$  given by

$$h(e) = L_f(e, w)(1) \quad \text{for } e \in F_{w(0)},$$

is a homeomorphism since it is obtained from the compositions of coordinate functions which are homeomorphisms. Hence  $\phi$  is a map from  $\Omega(O, r_o)$  into  $G$ .  $\square$

**Definition 4.4.** For any space  $O$  with fixed point  $r_o \in O$ , we can define a *conical map*  $\psi : O \longrightarrow \Omega(S(O), r_o)$  as follows:

For  $x \in O$ , let  $w_0(x)$  be path between in a cone  $S_0(O)$  from equivalent class  $[(x, 1/2)]$  into  $[(r_o, 1/2)]$  and let  $w_1(x)$  be path between in a cone  $S_1(O)$  from  $[(x, 1/2)]$  into  $[(r_o, 1/2)]$ . Define the *conical map*  $\psi : O \longrightarrow \Omega(S(O), r_o)$  by

$$\psi(x) = \overline{w_1(x)} \star w_0(x) \quad \text{for } x \in O,$$

where  $\Omega(S(O), r_o) := \Omega(S(O), [(r_o, 1/2)])$ .

To prove the equivalently between Theorem 1.2 and the Dold's theorem we will rephrase Theorem 1.2 for two fibrations over a common suspension base.

**Remark 4.5.** Let  $\gamma_1 = [E_1, f_1, S(O), F_{r_o}^1, G_1]$  and  $\gamma_2 = [E_2, f_2, S(O), F_{r_o}^2, G_2]$  be two fibrations over a common suspension base  $S(O)$  of a polyhedron space  $O$  with locally compact fibers  $F_{r_o}^1$  and  $F_{r_o}^2$ . In Figure 2, let

$$\mu_1 : (O, r_o) \longrightarrow (G_1, \mathbf{g}_1) \quad \text{and} \quad \mu_2 : (O, r_o) \longrightarrow (G_2, \mathbf{g}_2)$$

be characteristic maps of  $\gamma$  and  $\gamma'$ , respectively. Also let

$$i_1 : G_1 \longrightarrow H(F_{r_o}^1, F_{r_o}^1) \quad \text{and} \quad i_2 : G_2 \longrightarrow H(F_{r_o}^2, F_{r_o}^2)$$

be the inclusion maps. From Propositions 4.2 and 4.3, then Theorem 1.2 and the Dold's theorem can now be compared. Let  $g \in H(F_{r_o}^1, F_{r_o}^2)$  and  $\psi$  is the conical map. Hence

$$\begin{array}{ccccc} & & L(S(O), r_o) & & \\ & \nearrow \phi_2 & \uparrow \psi & \nwarrow \phi_1 & \\ G_2 & \xleftarrow{\mu_2} & O & \xrightarrow{\mu_1} & G_1 \\ \downarrow i_2 & & & & \downarrow i_1 \\ H(F_{r_o}^2, F_{r_o}^2) & \xleftarrow{T(f)=g \circ f \circ \overleftarrow{g} \quad \forall f \in H(F_{r_o}^1, F_{r_o}^1)} & & & H(F_{r_o}^1, F_{r_o}^1) \end{array}$$

Figure 2

Theorem 1.2 can be restated in terms of  $\phi_1$ ,  $\phi_2$ , and  $\psi$  as follows:

Two fibrations  $\gamma_1 = [E_1, f_1, S(O), F_{r_o}^1, G_1]$  and  $\gamma_2 = [E_2, f_2, S(O), F_{r_o}^2, G_2]$  are fiber homotopy equivalent if and only if there is  $g \in H(F_{r_o}^1, F_{r_o}^2)$  such that two maps

$$m(x) = g \circ i_1 \circ \phi_1[\psi(x)] \circ \overleftarrow{g} \quad \text{for } x \in O,$$

and

$$m'(x) = i_2 \circ \phi_2[\psi(x)] \quad \text{for } x \in O,$$

from  $O$  into  $H(F_{r_o}^2, F_{r_o}^2)$  are homotopic.

If it can be shown that  $\phi_1 \circ \psi \simeq \mu_1$  and  $\phi_2 \circ \psi \simeq \mu_2$ , then Theorem 1.2 and the Dold's theorem are equivalent. We will prove it in Theorem 4.7.

**Lemma 4.6.** Let  $\gamma = [E, f, O, F_{r_o}, G]$  be a fibration. Let  $O = O_1 \cup O_2$ ,  $r_o \in O_1 \cap O_2$ ,

$$\begin{aligned}\overline{\Delta}f &= \{(\beta, e) \in \Omega(O, r_o) \times F_{r_o} : \beta = w_2 \star w_1, w_i \in O_i^I (i = 1, 2), \\ &\quad \beta(1/2) = w_2(1) = w_1(0) \in O_1 \cap O_2\}\end{aligned}$$

and  $L_i$  be a lifting functions for fibration  $f|_{O_i}$ . If there are fiber homeomorphisms  $\epsilon_i : O_i \times F_{r_o} \longrightarrow f^{-1}(O_i)$ , then the map  $\overline{\phi} : \overline{\Delta}f \longrightarrow F_{r_o}$  given by

$$\overline{\phi}(\beta, e) = L_1[L_2(e, w_2)(1), w_1](1) \quad \text{for } (\beta, e) \in \overline{\Delta}f,$$

is homotopic to the  $L_f$ -function  $\Theta_{L_f}$ , where  $i = 1, 2$ .

**Proof.** We can define the lifting functions  $L_1$  and  $L_2$  for fibrations  $f|_{O_1}$  and  $f|_{O_2}$  by

$$L_i(e, w) = \epsilon_i[w(t), (\pi_2 \circ \epsilon_i^{-1})(e)] \quad \text{for } (e, w) \in \Delta f|_{O_i},$$

where  $i = 1, 2$ , respectively. Since  $L_f$  is lifting function for  $\gamma$ , then it is also a lifting function for  $f|_{O_1}$  and  $f|_{O_2}$ . Hence  $L_f \simeq L_1$  on  $\Delta f|_{O_1}$  and  $L_f \simeq L_2$  on  $\Delta f|_{O_2}$ . Define a map  $\tilde{\phi} : \overline{\Delta}f \longrightarrow F_{r_o}$  by

$$\tilde{\phi}(\beta, e) = L_f[L_f(e, w_2)(1), w_1](1) \quad \text{for } (\beta, e) \in \overline{\Delta}f.$$

Then  $\tilde{\phi} \simeq \overline{\phi}$  and by the homotopy  $H$  in proof of Lemma 3.9,  $\tilde{\phi} \simeq \Theta_{L_f}$ . Hence  $\Theta_{L_f} \simeq \overline{\phi}$ .  $\square$

**Theorem 4.7.** Let  $\gamma = [E, f, S(O), F_{r_o}, G]$  be a fiber bundle over suspension  $S(O)$  of a polyhedron space  $O$  with locally compact fiber  $F_{r_o}$  and admits a lifting function  $L_f$ . Also let  $\phi : \Omega(S(O).r_o) \longrightarrow G$  be a map given by

$$\phi(\beta)(x) = L_f(x, \beta)(1) \quad \text{for } \beta \in \Omega(S(O), r_o), x \in F_{r_o}.$$

Then  $\phi \circ \psi \simeq \mu$ , where  $\mu : (O, r_o) \longrightarrow (G, e)$  is the characteristic map of  $\gamma$  and  $\psi$  is the conical map.

**Proof.** In Lemma 4.6, put  $S(O) = S_0(O) \cup S_1(O)$ ,  $O_1 = S_0(O)$ , and  $O_2 = S_1(O)$ . It's clear that  $O = S_0(O) \cap S_1(O)$ . Now define maps

$$g_{ii} : S_i(O) \longrightarrow G \quad \text{by} \quad g_{ii}(x) = \mathbf{g} \quad \text{for } x \in S_i(O), (i = 0, 1),$$

$$g_{01} : O \longrightarrow G \quad \text{by} \quad g_{01}(x) = \mu(x) \quad \text{for } x \in O,$$

and

$$g_{10} : O \longrightarrow G \quad \text{by} \quad g_{10}(x) = [\mu(x)]^{-1} \quad \text{for } x \in O.$$

Let  $J = \{0, 1\}$  be a space with the discrete topology and let  $T \subset S(O) \times F_{r_o} \times J$  be the set defined by

$$T = \{(x, e, j) : x \in S_j(O), e \in F_{r_o}, j \in J\}.$$

Define an equivalent relation  $\equiv$  on  $T$  by

$$(x_1, e_1, j) \equiv (x_2, e_2, k) \iff x_1 = x_2 \quad \text{and} \quad g_{kj}(x_1)(e_1) = e_2,$$

where  $(x_1, e_1, j), (x_2, e_2, k) \in T$ .

Recall Proposition 4.1 that points of  $E$  are identified to the equivalent classes of all triples  $(x, e, j) \in T$ . Hence for  $j = 0, 1$ , the maps  $\epsilon_j : S_j(O) \times F_{r_o} \longrightarrow f^{-1}(S_j(O))$  given by

$$\epsilon_j(x, e) = [(x, e, j)] \quad \text{for } (x, e) \in S_j(O) \times F_{r_o},$$

denotes the equivalence class of the triple  $(x, e, j)$ .

Put  $r_o := [(r_o, 1/2)]$  in  $S(O)$  and  $e := [(r_o, e, i)]$ , where  $i = 0, 1$ . Then from Lemma 4.6, we have that for  $\beta \in \Omega(S(O), r_o)$ ,  $\beta = w_1 \star w_0$  for some  $w_0 \in [S_0(O)]^I$ ,  $w_1 \in [S_1(O)]^I$  and

$$\begin{aligned} L_1(e, w_0)(1) &= [(w_0(1), e, 0)], \\ L_2(e, w_1)(1) &= [(w_1(1), e, 1)] \\ &= [(w_1(1), \mu(w_1(1))(e), 0)]. \end{aligned}$$

Hence

$$\begin{aligned} \overline{\phi}(\beta, e) &= L_1[L_2(e, w_1)(1), w_0](1) \\ &= [(r_o, \mu(w_1(1))(e), 0)] \\ &= \mu(w_1(1))(e) \\ &= \mu(\beta(1/2))(e). \end{aligned}$$

Let  $\overline{\Omega}(S(O), r_o)$  be the projection of  $\overline{\Delta}f$  on  $\Omega(S(O), r_o)$  and  $\phi' : \overline{\Omega}(S(O), r_o) \longrightarrow G$  be a map given by

$$\phi'(\beta)(e) = \overline{\phi}(\beta, e) \quad \text{for } \beta \in \overline{\Omega}(S(O), r_o), e \in F_{r_o}.$$

Then by Lemma 3.9,  $\phi \simeq \phi'$  and

$$(\phi' \circ \psi)(e) = \phi'[\psi(e)] = \mu[\psi(e)(1/2)] = \mu(e),$$

Thus  $\phi' \circ \psi = \mu$ . Hence  $\phi \circ \psi \simeq \mu$ . Therefore Theorem 1.2 and the Dold's theorem are equivalent.  $\square$

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